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Profit Maximization and the Extinction of Animal Species

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In this paper I construct and analyze a simple mathematical model for the commercial exploitation of a natural animal population. The model takes into account the response of the population to harvesting pressure, the increasing harvesting costs associated with decreasing population levels, and the preference of the harvesters for present over future revenues. The principal conclusion of the analysis is that, depending on certain easily stated biological and economic conditions, extermination of the entire population may appear as the most attractive policy, even to an individual resource owner.

I. Introduction

Historical developments in the exploitation of many biological populations have followed a characteristic pattern. First comes a stage of expanding harvests, frequently quite rapid and often following some technological advance. As the exploitation increases, fears are expressed for the survival of the population and the associated industry. Conservation measures are considered and sometimes adopted with success. In other cases, action is taken too late and the industry collapses. In a few cases, species have been completely exterminated, or reduced to a population level incapable of surviving.

The study and practice of resource management is generally based on some form of bioeconomic modeling of the resource in question, the most extensive literature being devoted to fisheries management. Beginning with the paper of H. S. Gordon (1954), this literature has been particularly effective in analyzing the important consequences of "common property" conditions of exploitation.[1] The conclusion drawn from this analysis is that

[1] See Scott (1955), Zellner (1962), Turvey (1964), Christy and Scott (1965), Bachmura (1971), and Plourde (1971).

such conditions lead to economic inefficiency, and, in some cases, to over-exploitation in the biological sense, with yields and population levels lower than for maximum sustained physical yield. Although the possibility of complete extermination of populations has been recognized, few analyses have taken it explicitly into consideration (Smith 1969; Bachmura 1971; Gould 1972). In the literature on fisheries, for example, it has often been assumed, usually tacitly, that extermination is either impossible or unprofitable. This may be valid for many fisheries, but certainly not for other animal resources.

Analyses of controlled fisheries have often been based on the concept of rent maximization, that is, maximization of net annual revenues (Scott 1955; Smith 1969). It has been noted that, if harvesting costs rise with decreasing population levels, a rent-maximizing policy will automatically lead to biological conservation, with an equilibrium population *in excess* of the population corresponding to maximum sustained yield.

It is perhaps more reasonable to suppose, however, that the "sole owner" of a resource population would in fact choose to maximize the *present value* of his harvest sequence, discounting future revenues at some fixed rate. Several authors have remarked that in this case the corresponding equilibrium population level is lower than the rent-maximization level and may be either higher or lower than the maximum-yield level (Zellner 1962; Plourde 1970, 1971; Quirk and Smith 1970). It is even possible that a zero equilibrium population would be optimal in this sense, and it is this possibility that I wish to analyze in this article.

The model to be presented here is at the same time a generalization and a simplification of many existing models. Biologically, it is a seasonal model based on the usual notion of a reproduction curve; economically, it is a partial equilibrium model with fixed price and with unit cost assumed to depend only on population size.

Dynamically, the model utilizes a discrete time scale and is consequently so simple that it can be analyzed rigorously using only techniques of elementary calculus. The case of extinction arises naturally as an "end-point maximum," so that conditions giving rise to it are easily characterized. Roughly stated, the following are shown to be both necessary and sufficient conditions for extinction under present-value maximization: (*a*) the discount (or time preference) rate sufficiently exceeds the maximum reproductive potential of the population, and (*b*) an immediate profit can be made from harvesting the last remaining animals (or from reducing the population to a level too low for survival).

It must be emphasized that the model used here is not a welfare model, so that no assertion can be made to the effect that extinction is *socially* optimal. To repeat, the conclusion is only that extinction may result from present-value maximization.

Do the results so described have any practical application? Although it

is probably impossible to answer this question with any degree of certainty, I shall argue later that the Antarctic whale fishery may have been subject to the phenomenon described in this article. But whether present-value maximization has in fact played a role in the extinction of species is a question of imputing motives. My analysis suggests only that it could play such a role, the more so the higher the private rate of discount utilized.

The case of common-property exploitation, by the way, can be considered as a special case of present-value maximization, in which an *infinite* discount rate applies. Thus our analysis indicates that extinction would result under common-property conditions if and only if an immediate profit could be made from harvesting the last remaining members of the population.

II. The Bioeconomic Model

Since the work of Lotka and Volterra it has been standard in theoretical biology to consider population models in which the growth rate of the population is determined by the population itself, as expressed by a differential equation $dx/dt = F(x)$. In this article I shall use instead a recursion formula

$$x_{k+1} = f(x_k), \tag{1}$$

with the interpretation that x_k represents the breeding population at the beginning of the kth breeding season. The function $f(x)$ is called the "reproduction function"; for simplicity, it will be assumed increasing, concave, and differentiable on an interval $0 \leq x \leq \bar{x} = f(\bar{x})$ (fig. 1). Clearly $\bar{x}$ represents the *natural equilibrium population* level.
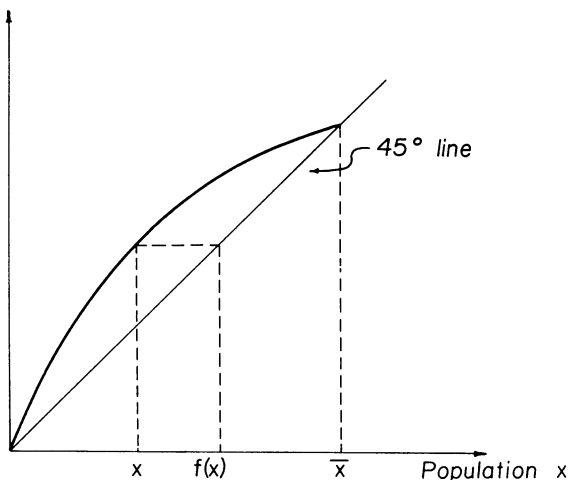

45° line

x   f(x)   x̄   Population x

Fig. 1

Equation (1) describes the "natural" behavior of the population being modeled. Suppose now that a seasonal harvest h_k is removed from the population x_k, the remaining population being left to breed. Then we have

$$x_{k+1} = f(x_k - h_k). \qquad (2)$$

The harvests h_k must satisfy

$$0 \leq h_k \leq x_k. \qquad (3)$$

Given the initial population x_1 and a known harvest sequence $\{h_k\}$ satisfying (3), the subsequent population levels are thus determined via (2).

Notice that any given breeding population x can be maintained by means of the harvest sequence

$$h_1 = x_1 - x; \qquad h_k = f(x) - x \qquad (k > 1). \qquad (4)$$

Next we need an economic component to our model. It will be supposed that (a) the revenue obtained from a harvest h is proportional to h through a fixed price $p > 0$, (b) the cost of obtaining a harvest h is determined by means of a marginal cost function $g(x)$ that represents the cost of harvesting one unit from a population of size $x + 1$. Therefore the cost of harvesting h units from a population of size $x + 1$ is given by

$$\sum_{k=x-h+1}^{x+1} g(k) \approx \int_{x-h}^{x} g(t) \, dt. \qquad (5)$$

The marginal cost function $g(x)$ is assumed continuous and nonincreasing (fig. 2). A typical example often used in fishery economics (Bradley 1970) is $g(x) = c/(x + 1)$. Further discussion of the marginal cost function is given in Section IV of this paper.
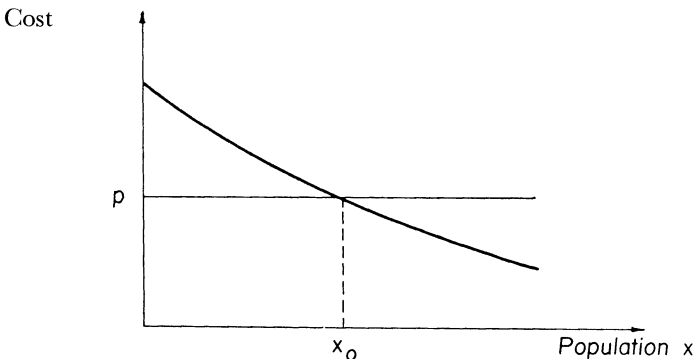

FIG. 2

It follows from (5) that the net revenue derived from a harvest h from an initial population $x + 1$ is given by

$$R(h; x) = ph - [G(x) - G(x - h)], \qquad (6)$$

where

$$G(x) = \int_0^x g(t) \, dt.$$

Let us consider first the problem of rent maximization, that is, maximization of $R(h; x)$ for a sustained yield $h = f(\xi) - \xi$ [so that $x = f(\xi)$]. Differentiating $R[f(\xi) - \xi; f(\xi)]$ with respect to the decision variable ξ, we obtain the necessary condition (unless $\xi = 0$ or $\bar{x}$):

$$f'(\xi) = \frac{M'(\xi)}{M'[f(\xi)]} \qquad (7)$$

where

$$M(x) = px - G(x),$$

and   (8)

$$M'(x) = p - g(x),$$

and this represents the marginal revenue—the revenue derived from harvesting the $(x + 1)$st member of the population. The economic significance of equation (7) will be discussed later in a more general setting (see eq. [15]).

There are three cost-price situations to consider (fig. 2).

Case 1: $p < g(x)$ for all x. Since no profit is possible, the optimal policy is given by $h = 0$, $\xi = \bar{x}$.

Case 2: $p = g(x_0)$ for some $x_0 > 0$; thus x_0 is the "zero profit" population level. From the assumptions on $f(x)$ and $g(x)$ it follows that

$$0 < \frac{M'(\xi)}{M'[f(\xi)]} < 1 \qquad \text{for } x_0 < \xi < \bar{x}. \qquad (9)$$

Consequently, since $f'(0) > 1$ and $f'(\bar{x}) < 1$, equation (7) has a solution ξ between x_0 and $\bar{x}$ (see fig. 3a). If ξ is unique (as we shall suppose for simplicity), it represents the optimal breeding population. Since $f'(x)$ is decreasing, we see that

$$\xi > x_{max}, \qquad (10)$$

where x_{max} is the breeding population corresponding to maximum sustained yield: $f'(x_{max}) = 1$.

Case 3: $p \geq g(0)$. In this case, we set $x_0 = 0$. Then (9) is again valid, and in addition [unless $p = g(0)$],

$$\lim_{\xi \to 0} \frac{M'(\xi)}{M'[f(\xi)]} = 1. \qquad (11)$$

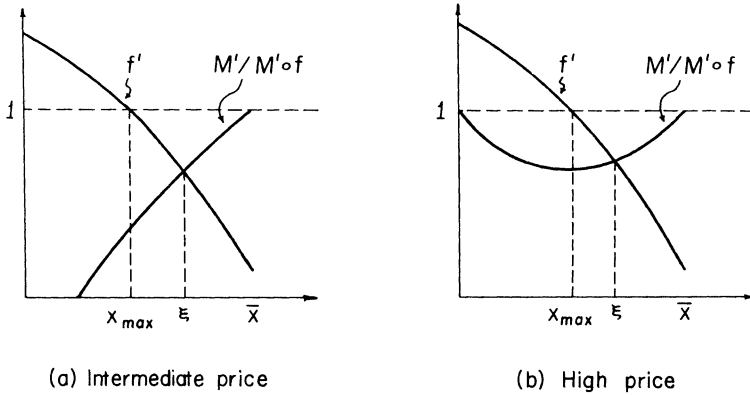(a) Intermediate price   (b) High price

Fig. 3

Equation (7) again has a solution ξ satisfying (10), so we reach the same conclusion as in case 2 (see fig. 3b). In particular, rent maximization never leads to extinction.

III. Maximization of Present Value

We now consider the problem of determining a harvest sequence $h = \{h_k\}$ that maximizes the present value:

$$P(h) = \sum_{k=1}^{\infty} R(h_k; x_k)\, \alpha^{k-1}. \tag{12}$$

Here $\alpha(0 < \alpha < 1)$ is the discount factor and the x_k is given by equation (2). Also the h_k must satisfy (3).

In order to simplify the analysis, we make the following assumption. Suppose that an optimal stock level x (to be determined) is chosen, and that the stock is reduced to this level by the first harvest and retained at the same level by all future harvests. (It will be proved in the Appendix that the policy h that maximizes the net present value $P(h)$ does behave in this manner.)

If x_1 denotes the initial stock level, we therefore have $h_1 = x_1 - x$, $h_2 = h_3 = \cdots = f(x) - x$. The present value of this harvest sequence is given by

$$P = p[x_1 - x] - [G(x_1) - G(x)]$$

$$+ \sum_{2}^{\infty} \alpha^{k-1} \left(p[f(x) - x] - \{G[f(x)] - G(x)\} \right) \tag{13}$$

$$= \frac{1}{1-\alpha} \left(\alpha\{pf(x) - G[f(x)]\} - [px - G(x)] \right) + [px_1 - G(x_1)].$$

Therefore the optimal stock level $x = \hat{x}$ for sustainable yield is determined by the condition

$$\hat{x} \text{ maximizes } \alpha\{pf(x) - G[f(x)]\} - [px - G(x)] = \alpha M[f(x)] - M(x). \tag{14}$$

By differentiation we obtain the necessary condition

$$\alpha f'(\hat{x}) = \frac{M'(\hat{x})}{M'[f(\hat{x})]} \tag{15}$$

unless $\hat{x} = 0$ or $\hat{x} = \bar{x}$. This equation generalizes (7). It can be written in the equivalent form

$$\frac{\alpha}{1 - \alpha}\frac{d}{dx}\{M[f(x)] - M(x)\} = M'(x); \qquad x = \hat{x},$$

which has the following interpretation. The right side is simply the marginal increase in current rent produced by a unit harvest at the population level $x + 1$. The left side, on the other hand, is the discounted present value of the marginal increase in sustained future rent resulting from a unit increase in population at the level $x + 1$. Maximization of the total present value P (at an interior value $x = \hat{x}$) requires equality of these marginal values.

As before, there are three cases in the analysis of equation (15).

Case 1: $p < g(x)$ for all x. Again, obviously $h_k = 0$ for all k.

Case 2: $p = g(x_0)$, $x_0 > 0$. As in figure 3a [but with $\alpha f'(x)$ in place of $f'(x)$], we see that equation (15) has a solution $\hat{x}$ satisfying $x_0 < \hat{x} < \bar{x}$. Since there is no relation between x_0 and x_{max}, we cannot conclude that $\hat{x} > x_{max}$; as $\alpha \to 0$ we have $\hat{x} \to x_0$.

It can be argued[2] that common-property exploitation corresponds to the case $\alpha = 0$ of total discounting, in which case $\hat{x} = x_0$.

Case 3: $p > g(0)$. An additional phenomenon arises in this case. First if α is near 1, then equation (15) has a positive solution $\hat{x}$ as before. But if α is sufficiently small, equation (15) will have no solution, and consequently $\hat{x} = 0$.

This is the principal new result of our analysis. To repeat: If price always exceeds unit cost, and if the discount rate $(1/\alpha)$ is sufficiently large, then maximization of present value results in extermination of the resource. Conversely if either of these conditions is lacking, then extermination does not result from present-value maximization.

The question of how large the discount rate must be for this phenomenon to occur will be taken up in the next section.

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[2] In a nutshell the argument is this: as long as additional profit can be made, new harvesters will be attracted. But once the population reaches the zero profit level, harvesting (for that season) will cease.

IV. The Marginal Cost Function $g(x)$

We have assumed that the unit, or marginal, harvesting cost $g(x)$ increases as the population level x decreases. Some authors have expressed this more explicitly by supposing that a given "effort," E (taken to be proportional to cost C), produces a harvest proportional to EN from a population of size N. In our notation this means simply that

$$g(x) = \frac{\text{const}}{x + 1}. \tag{16}$$

The latter formula can be derived, as an approximation, from simple probabilistic considerations. Suppose there is a probability $\rho > 0$ of catching a particular animal in a given unit of time. If there are x animals, then the probability of catching at least one of them in the given unit of time equals $1 - (1 - \rho)^x = P_x$. The probability of making the first catch during the nth times interval is $Q_x^{n-1} P_x$, where $Q_x = 1 - P_x$. Hence the expected waiting time for the first catch is

$$E_x = \sum_1^\infty n Q_x^{n-1} P_x = P_x \frac{d}{dx} \sum_0^\infty Q_x^n = \frac{P_x}{(1 - Q_x)^2} = \frac{1}{P_x} = \frac{1}{1 - (1 - \rho)^x}.$$

If we suppose that the cost of catching one animal is proportional to the time taken to catch it, then

$$g(x) = \frac{c}{1 - (1 - \rho)^{x+1}}, \qquad c = \text{constant}. \tag{17}$$

The function (17) applies also to the case in which the animals form into x groups (herds, schools, etc.), and these groups are randomly dispersed. In case $\rho \ll 1$ the function (17) is approximately the same as (16).

Both functions (16) and (17) satisfy the hypothesis of the following theorem, in which $1/\alpha = 1 + i$.

Theorem: Suppose that $g'(x)$ is an increasing (negative) function, and $f'(x)$ a decreasing function. Then maximization of present value (a) does not lead to extinction if either $p < g(0)$ or

$$1 + i < f'(0); \tag{18}$$

(b) does lead to extinction if $p \geq g(0)$ and

$$1 + i > [f'(0)]^2. \tag{19}$$

Proof: First, if $p < g(0)$ we know that $\hat{x} > x_0 > 0$. Also if $p \geq g(0)$, then since (18) means that $\alpha f'(0) > 1$, we see from figure 3b that $\hat{x} > 0$.

Conversely, suppose (19) holds, and $p \geq g(0)$.

Since $[M'(x)]/M'[f(x)] = [p - g(x)]/\{p - g[f(x)]\}$ is an increasing function of p for $p > g(0)$, it is sufficient to treat the case $p = g(0)$. In this case we have, by the generalized mean value theorem,

$$\frac{M'(x)}{M'[f(x)]} = \frac{g(0) - g(x)}{g[f(0)] - g[f(x)]} = \cdot \frac{g'(\xi)}{g'[f(\xi)]f'(\xi)}$$

$$> \frac{1}{f'(\xi)} > \frac{1}{f'(0)} \qquad (0 < \xi < x).$$

Since $f'(x)$ is decreasing, (19) now implies that equation (15) has no non-zero solution (see fig. 3b). This completes the proof.

We remark that (19) is the best possible result in case $p = g(0)$, but grows progressively less precise as the price rises above this level.

For the case of the Antarctic blue whale population, the Committee of Three Scientists appointed by the International Whaling Commission (1964) estimated a maximum reproductive potential of 10 percent per annum. Consequently an annual discount rate $i > (1.1)^2 - 1 = 21$ percent would suffice to cause the whalers to prefer extinction to conservation of the whales.

It is certainly true, on the other hand, that the whale stocks were common property. The fact remains that the International Whaling Commission was unable, for whatever reason, to adopt regulations necessary to prevent near complete extermination of the blue whale. Unfortunately, the intransigence of just one nation would be sufficient to thwart any efforts to reach an effective agreement. The degree of official interest in conserving the whales may perhaps be indicated by noting that each member nation contributed just £150, and later £250, per annum to the commission, which was charged with regulating an industry which could have produced an estimated annual revenue of from \$20 million to \$65 million on a sustained basis. (This refers to the blue whales alone.)[3]

Appendix

Derivation of the Optimal Policy

In order to characterize the optimal harvesting policy h, it is necessary to assume that the equation (15) for $\hat{x}$ has, at most, one solution. Thus $\hat{x}$ will denote this solution if it exists, and we will define $\hat{x} = 0$ otherwise.

It follows that for all x we have

$$\alpha f'(x) \begin{cases} > M'(x)/M'[f(x)] & \text{for } x < \hat{x} \\ < M'(x)/M'[f(x)] & \text{for } x > \hat{x}. \end{cases} \tag{20}$$

In this Appendix we show that under this condition the optimal policy $\hat{h} = \{\hat{h}_k\}$

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[3] Estimated maximum sustainable catch of blue whales: 6,000 (International Whaling Commission, 1964); estimated average market value per whale: \$3,600 (Europe), \$11,250 (Japan) (Small 1971, p. 42).

is just that sequence of harvests that leads as rapidly as possible to the equilibrium stock level $\hat{x}$. If x_1 (the original stock size) is $> \hat{x}$, this means that $\hat{h}_1 = x_1 - \hat{x}$, whereas if $x_1 < \hat{x}$, then $\hat{h}_1 = 0$. Thus, in general

$$\hat{h}_1 = \max (x_1 - \hat{x}, 0). \qquad (21)$$

Conversely, if (21) has been proved, then the entire sequence $\hat{h}$ is completely determined, since once $\hat{h}_1$ is known we have $x_2 = f(x_1 - \hat{h}_1)$, so that, since the sequence $\{\hat{h}_k\}$ must be optimal for $k \geq 2$, $\hat{h}_2 = \max (x_2 - \hat{x}, 0)$, and so on. Consequently we merely need to prove (21).

We give an indirect proof of (21). First, by a weak-compactness argument we show that there must exist a sequence $\hat{h}$, satisfying the necessary "admissibility" conditions (3), and maximizing the net present value $P(h)$. (Readers not familiar with functional analysis can simply skip this short proof and take the existence of $\hat{h}$ for granted.) Then we show that if $\hat{h}_1$ does not satisfy (21), a modified sequence h' can be found for which $P(h') > P(\hat{h})$. Since this is a contradiction, the proof is complete.

Lemma 1: There exists a sequence $\hat{h}$ such that $P(\hat{h}) = \sup_h P(h)$, where the supremum is taken with respect to all sequences h satisfying condition (3).

Proof: Let l_∞ denote the Banach space consisting of all real, bounded sequences $h = \{h_k\}$, with the norm $\|h\| = \sup |h_k|$. It is easy to verify from the assumed continuity of the function f that the set H of sequences satisfying (3) is a closed, bounded subset of l_∞, and hence H is weak*-compact.

The functional $P(h)$, even though it is nonlinear, is easily seen to be weak*-continuous on bounded sets in l_∞. Therefore $P(h)$ must assume a maximum value on H. QED.

Henceforth let $\hat{h} = \{\hat{h}_k\}$ denote an optimal harvest policy, and $\{\hat{x}_k\}$ the corresponding sequence of stock levels as determined by (2). (We will not know that $\hat{h}$ is uniquely determined until the proof of (21) is complete).

We next derive two lemmas leading to the proof of (21). For the proof of each lemma, we will suppose that an optimal sequence $\hat{h}$ does not satisfy some desired condition, and in each case a "modified" sequence $h' = \{h_k'\}$ will be constructed from $\hat{h}$ by altering only the value of $\hat{x}_2$, so that $x_k' = \hat{x}_k$ $(k \neq 2)$. Therefore we will have $h_i' = \hat{h}$ for $i \geq 3$, so that

$$\begin{aligned}
P(h') - P(\hat{h}) &= ph_1' - [G(x_1) - G(x_1 - h_1')] \\
&\quad + \alpha\{ph_2' - [G(x_2') - G(x_2' - h_2')]\} \\
&\quad - (p\hat{h}_1 - [G(x_1) - G(x_1 - \hat{h}_1)] \\
&\quad + \alpha\{p\hat{h}_2 - [G(\hat{x}_2) - G(\hat{x}_2 - \hat{h}_2)]\}) \qquad (22) \\
&= \{p(x_1 - \hat{h}_1) - G(x_1 - \hat{h}_1) - [p(x_1 - h_1') \\
&\quad - G(x_1 - h_1')]\} - \alpha\{p\hat{x}_2 - G(\hat{x}_2) \\
&\quad - [px_2' - G(x_2')]\} \\
&= M(x_1 - \hat{h}_1) - M(x_1 - h_1') \\
&\quad - \alpha[M \circ f(x_1 - \hat{h}_1) - M \circ f(x_1 - h_1')],
\end{aligned}$$

where $M \circ f(x) = M[f(x)]$. In each case, the latter expression will be seen to be positive, a contradiction implying the desired property for $\hat{h}$.

Since $f(x)$ is an increasing function, it is clear that no harvest that reduces the stock to a level below the zero profit level x_0 can be optimal. Therefore

$$x_1 - \hat{h}_1 \geq \min (x_1, x_0), \qquad (23)$$

and in particular, $\hat{h}_1 = 0$ in case $x_1 < x_0$.

In the remaining lemmas we use the following inequality:

$$\alpha[M \circ f(b) - M \circ f(a)] > M(b) - M(a) \text{ if } a < b \leq \hat{x} \text{ and } f(a) \geq x_0. \quad (24)$$

This is easily verified by noting that $M'[f(x)] > 0$ if $f(x) > x_0$, so that (20) implies

$$\alpha[M \circ f(b) - M \circ f(a)] = \alpha \int_a^b (M \circ f)'(x)\, dx$$

$$> \int_a^b M'(x)\, dx = M(b) - M(a).$$

In the same way, we also obtain

$$\alpha[M \circ f(b) - M \circ f(a)] < M(b) - M(a) \text{ if } \hat{x} \leq a < b \text{ and } f(a) \geq x_0. \quad (25)$$

Lemma 2: $x_1 - \hat{h}_1 \geq \min(x_1, \hat{x})$.

Proof: Suppose instead that $x_1 - \hat{h}_1 < \hat{x} \leq x_1$. We determine our modified sequence h' in this case by setting $x_2' = f(\hat{x})$. Then (22) yields

$$P(h') - P(\hat{h}) = M(x_1 - \hat{h}_1) - M(\hat{x}) - \alpha[M \circ f(x_1 - \hat{h}_1) - M \circ f(\hat{x})]$$

$$= \alpha[M \circ f(\hat{x}) - M \circ f(x_1 - \hat{h}_1)] - [M(\hat{x}) - M(x_1 - \hat{h}_1)],$$

and this is positive, according to (24), since $x_1 - \hat{h}_1 < \hat{x}$.

If $x_1 - \hat{h}_1 < x_1 < \hat{x}$, we simply take $x_2' = f(x_1)$, and proceed as before.   QED.

It follows from lemma 2 that

$$\hat{h}_1 = 0 \qquad \text{if } x_1 < \hat{x}. \quad (26)$$

Lemma 3: $x_1 - \hat{h}_1 \leq \hat{x}$.

Proof: This result is slightly more difficult to prove. As in the preceding lemma, the modified sequence h' will be obtained from $\hat{h}$ by altering a single value $\hat{x}_k$, but not necessarily $\hat{x}_2$.

Suppose now that $x_1 - \hat{h}_1 > \hat{x}$, that is $\hat{x}_2 > f(\hat{x})$. Now either $\hat{x}_3 < f(\hat{x}_2)$, or $\hat{x}_3 = f(\hat{x}_2)$. In the former case we can choose $\varepsilon > 0$ such that if $x_2' = \hat{x}_2 - \varepsilon$, then $x_2' > f(\hat{x})$ and $f(x_2') > \hat{x}_3$. It follows that the corresponding sequence h' is admissible, and from (25), that $P(h') > P(\hat{h})$.

This construction will not work if $\hat{x}_3 = f(\hat{x}_2)$, but in this case either $\hat{x}_4 < f(\hat{x}_3)$ or $\hat{x}_4 = f(\hat{x}_3)$. In the former case we can choose $x_3' = \hat{x}_3 - \varepsilon$ such that $x_3' > f(\hat{x})$ and $f(x_3') > \hat{x}_4$. Letting $x_k' = \hat{x}_k$ for $k \neq 3$, we obtain an admissible sequence h' satisfying $P(h') > P(\hat{h})$.

This process can be continued inductively as far as necessary. However, it is impossible that $\hat{x}_{k+1} = f(\hat{x}_k)$ for all $k \geq 2$, since the sequence $\{\hat{h}_1, 0, 0, \ldots\}$ is certainly not optimal in case $\hat{h}_1 < x_1 - \hat{x}$. Hence the above process eventually leads to a contradiction. QED.

Now, if $x_1 \leq \hat{x}$, then $\hat{h}_1 = 0$ by (26), and if $x_1 > \hat{x}$, then $x_1 - \hat{h}_1 = \hat{x}$ by lemmas 2 and 3. This proves (21) completely, and also shows that the optimal policy $\hat{h}$ is uniquely determined.

To conclude, let me describe briefly three interesting variations of the basic model which can be easily analyzed by essentially identical arguments.

A. Finite Time Horizon

Instead of an infinite sequence $\{h_k\}$ of future harvests, suppose we consider only sequences of fixed length N. Then the solution to the optimization problem is identical with the solution described here, except for the final harvest h_N, which must obviously be given by $h_N = x_N - x_0$ (assuming this is positive).

B. Limited Market

Suppose there is a fixed limit M to the market for the animal resource. Then condition (3) is supplemented by the inequalities, $h_k \leq M$ for all k. If $x_1 > \hat{x}$ we will then simply have $\hat{h}_1 = \min (x_1 - \hat{x}, M)$, and of course $\hat{h}_1 = 0$ in case $x_1 < \hat{x}$.[4] Thus the entire sequence $\hat{h}$ is again determined. As before, it is just the sequence that reaches the optimal stock level x in the shortest possible time, with the exception that if $M < f(\hat{x}) - \hat{x}$ then the optimal level will not be reached at all (Clark 1972).

C. An Absolute Conservation Standard

Suppose that no harvest is permitted to reduce the stock below some fixed lower limit $x = L$. If we assume that equation (15) has at most one solution $\hat{x} \geq L$, and define $\hat{x} = L$ in case it has no solution, then the foregoing theory remains valid. Any solutions of (15) less than L are simply ruled out by the lower limitation on the stock level.

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[4] The latter case requires modification if $M < f(\hat{x}) - \hat{x}$ (see Clark 1972).